

GENERALIZED HOPFIAN PROPERTY, MINIMAL HAKEN MANIFOLD, AND J. SIMON'S CONJECTURE FOR 3-MANIFOLD GROUPS

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ABSTRACT. We address a conjecture that π_1 -surjective maps between closed aspherical 3-manifolds having the same rank on π_1 must be of non-zero degree. The conjecture is proved for Seifert manifolds, which is used in constructing the first known example of minimum Haken manifold. Another motivation is to study epimorphisms of 3-manifold groups via maps of non-zero degree between 3-manifolds.

Section 1. Introduction and some examples.

Let M and N be closed 3-manifolds and $f : M \rightarrow N$ a map of non-zero degree, then the image of f_* is a subgroup of finite index in $\pi_1(N)$. If M and N are aspherical, any homomorphism $\phi : \pi_1(M) \rightarrow \pi_1(N)$ determines a unique map $f : M \rightarrow N$ up to homotopy such that $f_* = \phi$. It seems natural to ask when there exists $f : M \rightarrow N$ of non-zero degree given a homomorphism ϕ surjecting $\pi_1(M)$ on a subgroup of finite index in $\pi_1(N)$? There are elementary constructions of examples (see below) that show in general that the answer is no. Before discussing some examples we make the following definition.

Definition 1.1. A map $f : M \rightarrow N$ between 3-manifolds is π_1 -surjective (resp. π_1 -finite-index) if $f_* : \pi_1(M) \rightarrow \pi_1(N)$ is surjective (resp. the image of f_* is a subgroup of finite index).

Recall that if M is an n -manifold, the rank of $\pi_1(M)$ (or simply by abuse just M) is the minimal cardinality of a generating system for $\pi_1(M)$.

Let us first have a look of the situation in dimension 2 which is quite simple. Throughout the paper Σ_k will denote a closed orientable surface of genus k .

Example 1.1. It is not difficult to see that there is a π_1 -surjective map $f : \Sigma_l \rightarrow \Sigma_k$ which is of degree zero when $l \geq 2k$.

On the otherhand, if $f : \Sigma_l \rightarrow \Sigma_k$ is a π_1 -surjective map with $0 < l < 2k$, then we claim that f is of non-zero degree. The proof of this result is direct. Choose a 1-skeleton of Σ_k to be a one point wedge of $2k$ circles $\mathcal{V} = \vee C_i$. Fix a point x_i on C_i . If

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f is of degree zero, then the image of f can be deformed into \mathcal{V} . We assume therefore that this is the case. Since f_* is π_1 -surjective, $f : \Sigma_l \rightarrow \mathcal{V}$ must be surjective. We may also assume that f is transverse to each x_i , $i = 1, 2, \dots, 2k$. So $f^{-1}(\cup x_i)$ is a set of essential circles. Partition $f^{-1}(\cup x_i)$ into sets G_1, \dots, G_h such that two components are in the same set if and only if they are parallel. For each G_j , find an annulus A_j containing G_j . Then squeeze each A_j to an arc a_j and the part $\Sigma_k \setminus \cup A_j$ to a point. The quotient Q will be a bouquet of h circles. Since $\mathcal{V} - \{x_i, i = 1, 2, \dots, 2k\}$ is contractible, the map $f : \Sigma_l \rightarrow \mathcal{V}$ factors through $q : Q \rightarrow \mathcal{V}$ which is still π_1 -surjective. It follows that $h \geq 2k$. In particular, there are at least $h \geq 2k$ disjoint essential non-separating non-parallel circles. By a well known argument in surface topology, we must have that the l , the genus of Σ_l , is at least $2k$. \square

Let us come back to dimension 3. The first example illustrates the aspherical assumption.

Example 1.2:

Let $f = e \circ p : S^2 \times S^1 \rightarrow S^2 \times S^1$, where p is a map which pinches $S^2 \times S^1$ to S^1 , and e identifies S^1 to a fiber $* \times S^1 \subset S^2 \times S^1$. Clearly f is of zero degree but π_1 -surjective. \square

The second example shows that if we do not require that the manifolds have the same rank, then the answer to the question is no.

Example 1.3:

We construct a map $f : \Sigma_{g+1} \times S^1 \rightarrow \Sigma_g \times S^1$ of zero degree which is π_1 -surjective. The map f is the composition of the following four geometric operations.

Project $\Sigma_{g+1} \times S^1$ to Σ_{g+1} .

Squeeze a suitable separating circle on Σ_{g+1} to a point in such a way that the quotient space is a one point union of a torus and Σ_g .

Squeeze the torus to a circle in such a way that the quotient space is a one point union of the circle and Σ_g .

Send Σ_g and the circle to a section $\Sigma_g \times *$ and the circle fiber of $\Sigma_g \times S^1$ respectively. \square

The third example has the same purpose as the second one, but the manifolds in this case are hyperbolic.

Example 1.4:

Let M be a closed hyperbolic 3-manifold whose fundamental group surjects the free group of rank 2. Such examples are easily constructed by doing hyperbolic surgery on a null-homotopic hyperbolic knot in $\Sigma_2 \times S^1$ [Section 3, BW]. Let $\phi_1 : \pi_1(M) \rightarrow \mathbb{Z} * \mathbb{Z}$ denote such a map. Let N be any hyperbolic 3-manifold such that $\pi_1(N)$ has two generators, then there is an epimorphism $\phi_2 : \mathbb{Z} * \mathbb{Z} \rightarrow \pi_1(N)$. If we choose N such that the volume of N is larger than the volume of M , then the map realizing the epimorphism $\phi = \phi_2 \circ \phi_1$ must be zero degree by the work of Gromov and Thurston, [T]. We remark that the volumes of hyperbolic 3-manifolds of rank 2 are unbounded. Briefly, it follows from work of Adams that the volumes of hyperbolic 2-bridge knot complements are unbounded. Doing large enough hyperbolic Dehn surgeries on these gives the required family, see [CR].

In fact it can be seen directly that the map realizing ϕ must be of zero degree since

such a map factors through a 1-dimensional complex. \square

As a consequence of these examples, we state the following more refined version of the question posed above.

Question 1.5. *Let M and N be closed aspherical 3-manifolds such that the rank of $\pi_1(M)$ equals the rank of $\pi_1(N)$. Assume, that $\phi : \pi_1(M) \rightarrow \pi_1(N)$ is surjective or whose image is a subgroup of finite index. Does ϕ determine a map $f : M \rightarrow N$ of non-zero degree?*

Note if M and N are homeomorphic and satisfy Thurston's geometrization conjecture, then a π_1 -surjective map $f : M \rightarrow M$ must be degree one. For since $\pi_1(M)$ is hopfian, f_* is surjective implies f_* is an isomorphism. Since M is aspherical f must be a homotopy equivalence, and so in particular, f is of degree one. Thus the question above is a kind of generalization of the Hopfian property: the condition "homeomorphic manifolds" is replaced by "manifolds of the same rank", the condition " π_1 -surjective" is replaced by " π -surjective" or " π_1 -finite-index", and the conclusion replace "degree one" by "non-zero degree". It is easy to construct examples to show that "non-zero degree" cannot be sharpened to "degree one", see the examples in Section 3.

One of the main results of this paper is to prove that for Seifert fibered 3-manifolds Question 1.5 has a positive answer (see Theorem 2.1 and Remark 2.4). In §4 we use this result to construct the first known example of a Haken 3-manifold which is minimal with respect to degree 1 mappings in Thurston's picture of 3-manifolds (Theorem 4.1). The manifold is a graph manifold built from the union of two trefoil knot complements. An orientable 3-manifold M is minimal if there is a degree one map $f : M \rightarrow N$ implies either $N = S^3$ or $M = N$. Usually it is difficult tell if a 3-manifold is minimal. We remark that all minimal Seifert manifolds are non-Haken [LWZ], and that the known minimal hyperbolic 3-manifolds are also non-Haken [RW], see [BW], [RW] and [LWZ] for a further discussion of such matters.

We were also motivated by the following posed by J. Simon.

[K. Problem 1.12]. *Let $G_K = \pi_1(S^3 - K)$ for a knot K in S^3 . Conjectures: if there is an epimorphism $\phi : G_K \rightarrow G_L$, then*

- (A) *rank $G_L > \text{rank } G_K$.*
- (B) *genus(L) \geq genus(K).*
- (C) *Given K , there is a number N_K such that any sequence of epimorphisms of knot groups $G_K \rightarrow G_{L_1} \rightarrow \dots \rightarrow G_{L_n}$ with $n \geq N_K$ contains an isomorphism.*
- (D) *Given K , there are only finitely many knot groups G for which there is an epimorphism $G_K \rightarrow G$.*

These conjectures have seen little progress. On the otherhand, more recently, questions similar to (C) and (D) have been raised for degree one maps and there are already several substantial results in this setting.

[K Problem 3.100]. *Let M be a closed orientable 3-manifold.*

- (A) *Are there only finitely many irreducible 3-manifolds N such that there exists a degree one map $M \rightarrow N$?*

(B) Does there exist an integer N_M such that if a sequence of degree one map $M \rightarrow M_1 \rightarrow \dots \rightarrow M_k$ with $k > N_{M_0}$, the sequence contains an homotopy equivalence

If one assumes Thurston's Geometrization conjectural picture of 3-manifolds, the answer for (B) is Yes if $k = \infty$ [Ro2]; the answer for (A) is Yes if the targets are hyperbolic [So], or the domain is non-Haken [RW], or the targets have finite π_1 [LMWZ].

Thus it seems natural to study the conjectures of J. Simon for closed orientable 3-manifolds (Question 1.6 in §3). We find that the positive answer for Question 1.5 are important for studying the conjectures. This will be addressed in §3.

Section 2. π_1 -surjective maps between aspherical Seifert manifolds.

Theorem 2.1. *Let M_1 and M_2 be closed orientable aspherical Seifert fiber spaces with the same rank and whose base orbifolds are orientable. Then any π_1 -surjective map $f : M_1 \rightarrow M_2$ is of non-zero degree.*

To prove Theorem 2.1, we will make use of [Ro1], in particular we refer the reader to [Ro1] for the definition of a *vertical pinch* and a *squeeze*, and *squeeze torus*. Call a squeeze is *vertical*, if in the squeeze torus, the squeezing circle meets the regular fiber exactly one point.

Also remember that any orientable Seifert manifold M with orientable base orbifold of genus g and with n singular fibers has a unique normal form $(g; b; \alpha_1, \beta_1; \dots; \alpha_n, \beta_n)$, where $0 \leq \beta_i \leq \alpha_i$, $i = 1, \dots, n$. The orbifold O_1 of M_1 will denoted by $(g; \alpha_1, \dots, \alpha_n)$. g is often omitted if $g = 0$.

We first need two lemmas.

Lemma 2.2 [Ro1, Lemma 3.5]. *Let $f : M \rightarrow N$ be a map between aspherical Seifert manifolds and $1 \neq f_*(h) \subset h'$, where M is closed and $\partial N \neq \emptyset$, h and h' are regular fibers of M and N respectively. Then either f admits a vertical squeeze, or f can be homotoped so that the image of f lies in a fiber of N .*

Lemma 2.3. *Suppose $f : F \rightarrow O$ is an orbifold branch covering, where F is a surface of genus g , and O is a orbifold, both are orientable and have non-positive Euler characteristic, then $\text{rank}(\pi_1(F)) \geq \text{rank}(\pi_1(O)) - 1$ if f is a double branched cover over 2-sphere and $\text{rank}(\pi_1(F)) \geq \text{rank}(\pi_1(O))$ otherwise.*

Proof. The proof is based on the results about the ranks of Fuchsian groups [ZVC, Theorem 4.16.1] and the Riemann-Hurwitz formula.

Suppose O has k singular points of index v_i , $i = 1, \dots, k$, with the underlying space of genus g' and the degree of f is n . Then we have

$$2 - 2g = n(2 - 2g' - \sum_{i=1}^k (1 - \frac{1}{v_i}))$$

For the case $g = 1$, the verification is direct, so we assume below that $g > 1$.

If $n = 2$ then all $v_i = 2$, $k = 2m$ and we have $2 - 2g = 2(2 - 2g' - m)$, i.e., $g = 2g' + m - 1$. Now $\text{rank}(\pi_1(F)) = 2g = 4g' + 2m - 2$ and the $\text{rank}(\pi_1(O))$ is at

most $2g' + 2m - 1$ if $g' > 0$ and is $2m - 1$ if $g' = 0$ by [ZVC, Theorem 4.16.1]. In any case the lemma follows.

If $n \geq 3$, then

$$2 - 2g \leq 3(2 - 2g' - \sum_{i=1}^k (1 - \frac{1}{v_i})) \leq 3(2 - 2g' - \frac{k}{2})$$

i.e., $2g \geq 6g' - 4 + \frac{3k}{2}$. If $g' > 0$, $2g \geq 2g' + \frac{3k}{2}$. But the rank of $\pi_1(O)$ is at most $2g' + k - 1$. If $g' = 0$, then we have $2g \geq -4 + \frac{3k}{2}$ if k is even and $g \geq -4 + \frac{3k}{2} + \frac{1}{2}$ if k is odd. The rank of $\pi_1(O)$ is at most $k - 1$. It follows that if $k \leq 5$, then $2g \geq k - 1$. If $k \leq 4$ we still have $2g \geq k - 1$ since we assume that $g > 1$. \square

Proof of Theorem 2.1. Suppose f is of zero degree. For clarity, the proof is divided up three steps.

Step (1) We prove the following:

Claim : $f(h)$ is homotopically non-trivial, where h is the regular fiber of M_1 .

Proof of Claim:

Let $M_1 = (g; b; a_1, b_1; \dots; a_k, b_k)$ and $G_1 = \pi_1(M_1)/\langle h \rangle$, where $\langle h \rangle$ is the cyclic group generated by regular fiber of M_1 . Let $r = \text{rank}(\pi_1(M_1)) = \text{rank}(\pi_2(M_2))$.

By [BZ, Theorem 1.1] and [ZVC, Theorem 4.16.1], one of the following cases holds.

(1) $\text{rank}(\pi_1(M_1)) > \text{rank}(G_1)$,

(2) $\text{rank}(\pi_1(M_1)) = \text{rank}(G_1)$ there is a set of generators of G_1 which realizes the rank and contains at least one torsion element,

(3) $\text{rank}(\pi_1(M_1)) = \text{rank}(G_1) = \text{rank}(G_1/T)$, where T is the normal subgroup normally generated by the torsion elements and G_1/T is a surface group.

If $f(h)$ is homotopically trivial, then $f_* : \pi_1(M_1) \rightarrow \pi_1(M_2)$ induces an epimorphism $\phi : G_1 \rightarrow \pi_1(M_2)$.

In Case (1) the Claim is clearly true.

In Case (2) the Claim is also true since $\pi_1(M_2)$ is torsion free.

In Case (3), f_* induces an epimorphism $\phi' : G_1/T \rightarrow \pi_1(M_2)$. Let $f' : F \rightarrow M_2$ be the map which realizes ϕ' . Since ϕ' is not injective, (otherwise ϕ' would be an isomorphism and $\pi_1(M_2)$ would be surface group), by the simple loop theorem for maps from a surface to a Seifert manifold [H], there are essential simple loops in the kernel of ϕ' . Assume first there is an essential non-separating simple loop, which we α , in the kernel. Then the map f' induces a map $f'' : F' \rightarrow M_2$, where F' is a complex obtained by squeezing F along α . It is easy to see that the rank of $\pi_1(F')$ is $r - 1$. We reach a contradiction. If all essential simple loops in kernel of ϕ' are separating, let α be a maximal family of non-parallel separating essential simple closed curves in kernel ϕ' . Again f' can factors through $f'' : F' \rightarrow M_2$, where F' is a complex of obtained by squeezing F along α , which is union of closed surfaces connected by arcs. Let S be a surface in F' . Due to the maximality of α , the restriction $f''|_S$ π_1 -injective, which must be either horizontal or vertical by [H]. If $f''|_S$ is horizontal, than $p_2 \circ f''| : S \rightarrow O(M_2)$ is an orbifold branched covering, where $p_2 : M_2 \rightarrow O(M_2)$ is the fiber map. But the rank of $\pi_1(S)$ is at most $r - 2$. This is also ruled out by Lemma 2.3. If $f''|_S$ is vertical

for each surface S of F' , then F' contains at most g such surfaces and each of them is a torus. Clearly the rank of $f''_*\pi_1(F')$ is at most $g + 1$, which is at most $r - 1$ (since $g > 1$ and $r \geq 2g$). Again we reach a contradiction.

Step (2)

We will factor $f : M_1 \rightarrow X \rightarrow M_2$, where the 2-dimensional complex X is a quotient of M with rank r_X .

Since $f(h)$ is homotopically non-trivial, a standard argument in 3-manifold topology shows that $f : M_1 \rightarrow M_2$ can be deformed to be a fiber preserving map (see [J] for example). Since a vertical pinch reduces the rank of π_1 , $f : M_1 \rightarrow M_2$ admits no vertical pinch. Suppose the mapping degree is zero. We can further deform the map so that the image $f(M_1)$ misses a regular fiber h' of M_2 . To see this, $f : M_1 \rightarrow M_2$ is fiber preserving. We can further deform f so that for each singular fiber of M_2 , its preimage consists of finitely many fibers of M_1 . Now remove all singular fibers of M_1 and their f -images, and remove all singular fibers of M_2 and their f -preimages. The restriction of f gives a proper map $f' : M'_1 \rightarrow M'_2$, which is fiber preserving map between circle bundles. Since f is assumed to be degree zero, f' is of zero degree. Since $f(h)$ is non-trivial, the induced proper map $\bar{f}' : F'_1 \rightarrow F'_2$ between base surfaces must be degree zero. Hence by \bar{f}' can be deformed so that its image misses a point of F'_2 . This deformation can be lifted to the bundle map f' whose image then misses a circle fiber in M'_2 . With this we reach the situation claimed above.

Now remove an open fibered neighborhood of h' , and denote the resulting manifold by N . Then we have a fiber preserving map $f : M_1 \rightarrow N$, where $\partial N \neq \emptyset$.

According Lemma 2.2 either $f : M_1 \rightarrow N$ admits a fiber squeeze along an incompressible vertical torus, or $f(M_1) \subset$ a fiber of N . Using this we can reformulate the above so that either $f : M_1 \rightarrow M_2$ admits a vertical squeeze along an incompressible vertical torus, or $f(M_1) \subset$ a fiber of M_2 .

Since f is π_1 -surjective, and M_2 is an closed aspherical Seifert fiber space, the situation that $f(M_1) \subset$ a fiber of M_2 cannot happen. Let \mathcal{T} be a maximal family of disjoint non-parallel incompressible tori along which f admits vertical squeeze. Let $X = \mathcal{Q} \cup \mathcal{A}$ be the space obtained after the squeezing, where \mathcal{Q} is a union of Seifert fiber spaces with the induced Seifert fibration, \mathcal{A} is a union of annuli and $\partial\mathcal{A}$ is a union of regular fibers of \mathcal{Q} . Then f induces a π_1 -surjective map $X \rightarrow M_2$, which we continue to denote by f .

Now all components of \mathcal{Q} are Seifert fibered spaces with the induced Seifert fibrations, so we may assume that Q_1, \dots, Q_{k_1} are Seifert manifolds of \mathcal{Q} which are not the trivial circle bundle over S^2 and $Q_{k_1+1}, \dots, Q_{k_1+k_2}$ are trivial circle bundle over S^2 . Clearly each Q_j , $j > k_1$, is $S^2 \times S^1$.

For each $j > k_1$, there is an annulus A in \mathcal{A} , with two components C_1 and C_2 of ∂A such that C_1 belongs to Q_j . Since C_1 is a regular fiber of $Q_j = S^2 \times S^1$, $Q_j \cup A$ has C_2 as a retractor, and hence we can send $Q_j \cup A$ to C_2 by this retract, then extend the map to whole $\mathcal{Q} \cup \mathcal{A}$. After k_2 such operations, we get a quotient space $X_1 = \mathcal{Q}_1 \cup \mathcal{A}_1$ where $\mathcal{Q}_1 = \{Q_1, \dots, Q_{k_1}\}$, and f induces a π_1 -surjective map $X_1 \rightarrow M_2$,

By the maximality of \mathcal{T} , each Q_i contains no squeeze torus for $f|_{Q_i}$, so we have that

$f(Q_i) \subset$ a fiber of M_2 , and consequently we have the following

Fact (*)

: each Q_i has base orbifold S^2 and has no more than 3 singular fibers (otherwise there will be a squeeze torus).

So $f : X_1 \rightarrow M_2$ induces a π_1 -surjective map $X = \mathcal{S} \cup \mathcal{A}_1 \rightarrow M_2$, \mathcal{S} is a union of k_1 circle.

Step (3)

We will show that $r_X < r$ and reach a contradiction.

If $g > 0$, there is a non-separating squeeze torus for f and clearly $r_X < r$.

Below we assume that $g = 0$. Then every squeeze torus is a separating torus.

Say that M_1 is of type I, if M_1 has normal form $(0; b; 2, 1; \dots; 2, 1; 2\lambda + 1, b_k)$, where $k \geq 4$ is even, and $\lambda > 1$, otherwise call M_1 is of type II. By [Theorem 1.1 BZ], the $r = k - 2$ if M_1 is of type I and $r = k - 1$ if M_1 is of type II.

Each component Q_i of \mathcal{Q} must have infinite fundamental group, otherwise $f(h)$ is an element of finite order, which must be trivial in $\pi_1(M_2)$, and this is forbidden by Step (1). In particular, each Q_i contains at least 2 singular fibers, $i = 1, \dots, k_1$, and Q_i contains exactly two singular fibers only if $Q_i = (0; 0; a, b; a, -b)$. Moreover if M_1 is of type I, then at least one Q_i contains 4 singular fibers (since $\lambda > 1$ and both $(0; b; 2, 1; 2\lambda + 1, b_2)$ and $(0; b; 2, 1; 2, 1; 2\lambda + 1, b_3)$ have finite fundamental groups), which is not possible by Fact*.

If M_1 is of type II, then $r = k - 1$ and $k \geq 3$, but

$$r_X \leq k_1 \leq \frac{k}{2} < k - 1 = r,$$

where the first \leq is due to the fact that every squeeze torus is separating and the second \leq is due to every Q_i contains at least two singular fibers. \square

Remark 2.4. In Theorem 2.1, the condition " f is π_1 -surjective" can be replaced by " f is π_1 -finite-index", and the condition "orbifolds are orientable" can be removed. For details see [Hu], where the proof is parallel to the proof above, but involves more complicated case by case argument.

3. On the conjectures of J. Simon on 3-manifold groups.

In this section we study the following questions.

Question 1.6. Let M_i be closed orientable aspherical 3-manifolds. Suppose there is an epimorphism $\phi : \pi_1(M_1) \rightarrow \pi_1(M_0)$.

(A) Is $\text{rank } \pi_1(M_1) > \text{rank } \pi_1(M_0)$?

(B) Is Heegaard genus of $M_1 \geq$ Heegaard genus M_0 ?

Moreover given M_0 .

(C1) Is there a number N_M such that any sequence of epimorphisms $\pi_1(M_0) \rightarrow \pi_1(M_1) \rightarrow \dots \rightarrow \pi_1(M_n)$ with $n \geq N_M$ contains an isomorphism?

(C2) Does any infinite sequence of epimorphisms $\pi_1(M_0) \rightarrow \pi_1(M_1) \rightarrow \dots \rightarrow \pi_1(M_n) \rightarrow \dots$ contain an isomorphism?

(D) Are there only finitely many M_i with the same first Betti number, or the same π_1 -rank, as that of M_0 , for which there is an epimorphism $\pi_1(M_0) \rightarrow \pi_1(M_i)$?

We remark that a positive answer for (B) of Question 1.6 implies a positive solution to the Poincare Conjecture. From Example 1.4 of The Introduction the answer to (D) is negative if we remove the condition on first Betti number or π_1 -rank on (D) of Question 1.6.

We describe first some examples of non-trivial π_1 -surjective maps between two 3-manifolds of the same rank, which give a negative answers of the (A) of Question 1.6. Clearly those examples are all of non-zero degrees.

Example 3.1. Let M be a Seifert manifold of normal form $(0; 0; 6, b_1; 5, b_2; 7, b_3)$. Let \mathbb{Z}_2 be a cyclic group acting on M such that it induces the identity on the base space and standard rotation on each regular fiber. Then one verifies that M/\mathbb{Z}_2 is a Seifert manifold with normal form $(0; 0; 3, b_1; 5, 2b_2; 7, 2b_3)$. Now

$$\pi_1(M) = \langle s_1, s_2, s_3, h \mid [s_j, h], s_1^6 h^{b_1}, s_2^5 h^{b_2}, s_3^7 h^{b_3}, s_1 s_2 s_3 \rangle$$

and

$$\pi_1(M/\mathbb{Z}_2) = \langle t_1, t_2, t_3, h' \mid [t_j, h'], t_1^3 h'^{b_1}, t_2^5 h'^{2b_2}, t_3^7 h'^{2b_3}, t_1 t_2 t_3 \rangle$$

The quotient map $p : M \rightarrow M/\mathbb{Z}_2$ is a branched covering of degree 2 and p_* sends $s_j \mapsto t_j$ and $h \mapsto h'^2$. Since $(2, b_1) = 1$, p_* is surjective. By [BZ] these manifolds have rank 2. \square

Examples 3.2. We now give some examples of π_1 -surjective non-zero degree maps between hyperbolic manifolds of the same π_1 ranks.

Let M be a closed orientable 3-manifold and $k \subset M$ be any hyperbolic fibered knot. Suppose the fiber F has genus g . Let M_n be the n -fold cyclic branched cover of M over the knot k . Then the rank of $\pi_1(M_n)$ is bounded by $2g+1$ for all n and M_n is hyperbolic when n is large. If $k|n$, then $M_n \rightarrow M_k$ is a branched cover, which is π_1 -surjective. So there are must be infinitely many π_1 -surjective branched covering $M_n \rightarrow M_k$ between hyperbolic 3-manifolds of the same ranks.

A well studied case is when M_n is the n -fold cyclic branched cover of the figure eight knot. Then for $n \geq 3$ the fundamental groups are all 2-generator—in fact they are the Fibonacci groups $F(2, 2n)$, which are all hyperbolic if $n \geq 4$. By abelianizing $F(2, 2n)$ we see that all M_n have first Betti number zero (see [MR] for example).

The next example gives the negative answer of (C1) of Question 1.6.

Example 3.3

(1) Let $M(n, k) = (0; 0; 2^k 3, b_1; 5, 2^{n-k} b_2; 7, 2^{n-k} b_3)$. Similar to Example 3.1, we have sequence of degree 2 branched covering $M(n, n) \rightarrow \dots \rightarrow M(n, 1) \rightarrow M(n, 0)$ of length $n+1$, which induces a sequence of epimorphisms of groups $\pi_1(M(n, n)) \rightarrow \dots \rightarrow \pi_1(M(n, 1)) \rightarrow \pi_1(M(n, 0))$ of rank 2. Let M be $\Sigma_2 \times S^1$. Clearly $\pi_1(M)$ surjects onto $Z * Z$, then we have the sequence of epimorphisms

$$\pi_1(M) \rightarrow \pi_1(M(n, n)) \rightarrow \dots \rightarrow \pi_1(M(n, 1)) \rightarrow \pi_1(M(n, 0))$$

of length $n + 2$, where n can be arbitrarily large.

Moreover if we choose b_1, b_2, b_3 such that the Euler number of $M(n, n)$ is non-zero. Since each $M(N, k)$ has infinite π_1 and is the image of $M(n, n)$ under non-zero degree map, the Euler number of $M(n, k)$ is non-zero [Theorem 2, W]. It follows $M(n, k)$ has neither horizontal or vertical incompressible surface, and therefore all $M(n, k)$ are non-Haken [J].

(2) Let M_n be the n -fold cyclic branched covering of S^3 over figure eight knot as in the end of Example 3.2. Then we have sequence of branched coverings of hyperbolic rational homology spheres $M_{4k} \rightarrow \dots \rightarrow M_8 \rightarrow M_4$ of length l which induces a sequence of epimorphisms of groups $\pi_1(M_{4k}) \rightarrow \dots \rightarrow \pi_1(M_8) \rightarrow \pi_1(M_4)$ with rank 2. Let M be a hyperbolic 3-manifold with $\pi_1(M)$ surjecting $\mathbb{Z} * \mathbb{Z}$ (as in Example 1.4). Then we have the sequence of epimorphisms

$$\pi_1(M) \rightarrow \pi_1(M_{4k}) \rightarrow \dots \rightarrow \pi_1(M_8) \rightarrow \pi_1(M_4)$$

of length $l + 1$, l can be arbitrarily large. \square

The next result gives a partial positive answer of (C2) of Question 1.6.

Theorem 3.4. *Given M_0 , and a sequence M_i of closed orientable aspherical Seifert manifolds with epimorphisms $\pi_1(M_0) \rightarrow \pi_1(M_1) \rightarrow \dots \rightarrow \pi_1(M_n) \rightarrow \dots$, this sequence contains an isomorphism.*

Proof. By passing an infinite subsequence, we may assume that all groups in the sequence have the same rank (each epimorphism in the subsequence is the composition of epimorphisms involved). Then each epimorphism $\phi_i : \pi_1(M_i) \rightarrow \pi_1(M_{i+1})$ in the sequence can be realized by a map $f_i : M_i \rightarrow M_{i+1}$ of non-zero degree by Theorem 2.1. Moreover the Seifert fibrations of the M_i 's can be arranged so that each f_i is a fiber preserving. Let O_i be the orbifold of M_i , then $\chi(O_i) \leq 0$ and we have the induced sequence of epimorphisms

$$\pi_1(O_0) \rightarrow \pi_1(O_1) \rightarrow \dots \rightarrow \pi_1(O_n) \rightarrow \dots$$

of Fuchsian groups. We therefore have a decreasing sequence

$$\chi(M_0) \leq \chi(M_1) \leq \dots \chi(\pi_1(M_n)) \leq \dots$$

The $\{-\chi(O)\}$ form a well-order subset of reals, where O runs over compact orbifolds, $\chi(O_k) = \chi(O_{k+1})$ for k larger than a given N ([Ro2, Lemmas 2.5 and 2.6] for details). Since there are at most finitely many orbifolds O with given χ , by passing an infinite sequence, we may assume that all O_i are the same.

Let $O_i = (g; \alpha_1, \dots, \alpha_n)$. Then $M_i = (g; b_i; \alpha_1, \beta_{1,i}; \dots; \alpha_n, \beta_{n,i})$.

Since $0 < \beta_{l,i} < \alpha_l$ for $l = 1, \dots, n$, by passing a further subsequence, we may assume that $\beta_{l,i} = \beta_l$, and finally we get $M_i = (g; b_i; \alpha_1, \beta_1; \dots; \alpha_n, \beta_n)$. Moreover we may assume that all $b_i \neq 0$. Note that by [p. 680 of LWZ], all M_i have the same first Betti number and the torsion part of $H_1(M_i, \mathbb{Z})$ is unbounded if b_i unbounded. Since epimorphisms on π_1 induce epimorphisms on first homology groups, it follows that b_i 's are bounded.

Now we have $b_i = b_j$ for some i, j , then $M_i = M_j$ and by the hopfian property of Seifert manifold groups, the epimorphism $\pi_1(M_i) \rightarrow \pi_1(M_j)$ is an isomorphism. Then in the sequence above there must be an isomorphism. Theorem 3.4 follows. \square

We have seen that Theorem 2.1 plays important roles for the proof Theorem 3.4. If the answer of Question 1.5 is also YES for hyperbolic 3-manifolds, this will lead to a positive answer for (C2) and (D) for hyperbolic 3-manifolds.

Proposition 3.5. *Suppose Question 1.5 has a positive answer for hyperbolic 3-manifolds. Then for a given closed orientable hyperbolic 3-manifold M_0 ,*

(1) any infinite sequence of epimorphisms $\pi_1(M_0) \rightarrow \pi_1(M_1) \rightarrow \dots \rightarrow \pi_1(M_n) \rightarrow \dots$ contains an isomorphism, where all M_i are closed orientable hyperbolic 3-manifolds.

(2) there are only finitely many closed orientable hyperbolic 3-manifolds M_i with the same π_1 -rank as that of M_0 , for which there is an epimorphism $\pi_1(M_0) \rightarrow \pi_1(M_i)$.

Proof. (1) By passing an infinite subsequence we may assume all $\pi_1(M_i)$ have the same rank. Since we assumed that Question 1.5 has a positive answer for hyperbolic 3-manifolds, this sequence is realized by a sequence of non-zero degree maps

$$M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_n \rightarrow \dots$$

The rest of the proof is now standard. Since all maps $f_i : M_i \rightarrow M_{i+1}$ in the sequence are of non-zero degree, by Gromov's Theorem [Chapter 6, Th], $v(M_i) \geq v(M_{i+1})$, where $v(M_i)$ is the hyperbolic volume of M_i . By Thurston-Jørgenson's Theorem [Chapter 6, Th], $v(M_k)$ must be a constant when k is larger than a given integer N . Then by Gromov-Thurston's Theorem [Chapter 6, Th], f_k is homotopic to a homeomorphism, $k > N$, so f_{k*} is an isomorphism

For (2) since we again assume that Question of 1.5 has a positive answer for hyperbolic 3-manifolds, each $\phi_i : \pi_1(M_0) \rightarrow \pi_i(M_i)$ can be realized by a map of non-zero degree. By Soma's theorem [So], there are only finitely many such M_i . \square

We also note the following partial positive answer of (D) of Question 1.6 follows easily from the methods of [RW].

Theorem 3.6. *Suppose M is a non-Haken hyperbolic 3-manifold. Then there are only finitely many closed orientable hyperbolic 3-manifolds M_i for which there is an epimorphism $\pi_1(M) \rightarrow \pi_1(M_i)$.* \square

Section 4. A minimal Haken manifold

Let E be the complement of trefoil knot with m the meridian and l the longitude. E has a unique Seifert fibration with two singular fiber of indices 2 and 3, over the disc. Via this Seifert structure, we have a presentation

$$\pi_1(E) = \langle a, b, c, t \mid a^2t, b^3t, abc \rangle$$

where t is the regular Seifert fiber. Let E_1 and E_2 be homeomorphic to E with meridian and longitudes (m_i, l_i) , $i = 1, 2$. Now glue E_1 to E_2 via a homeomorphism $h : \partial E_1 \rightarrow \partial E_2$ such that $h(l_1) = m_2$ and $h(m_1) = l_2^{-1}$. Let M denote the resulting manifold, which is a closed graph manifold. The main theorem of this section is:

Theorem 4.1. *M is a minimal closed Haken 3-manifold among all 3-manifolds satisfying Thurston's geometric conjecture.*

We begin the proof by collecting some elementary facts.

Lemma 4.2. (1) *For any representation $\phi : \pi_1(E) \rightarrow SL(2, \mathbb{C})$, if $\phi(t) \neq 1$, then the image $\phi(\pi_1(E))$ is a cyclic group $\langle \lambda \rangle$. Moreover, we must have $\phi(a) = \lambda^{-2}$, $\phi(b) = \lambda^{-3}$, $\phi(c) = \lambda^5$, and $\phi(t) = \lambda^6$.*

(2) *In $\pi_1(T)$, where $T = \partial E$, we have $m = tc^{-1}$ and $l = t^{-5}c^6$. (Equivalently, $t = 6m + l$ and $c = 5m + l$.) Hence $h(t_1^{-5}c_1^6) = t_2c_2^{-1}$ and $h(t_1c_1^{-1}) = t_2^5c_2^{-6}$.*

(3) *M is an integral homology 3-sphere.*

(4) *the only 2-sided incompressible surface is the incompressible torus T , which separates M into E_1 and E_2 .*

Proof. The main part of (1) follows from [M, Prop. 3] and the fact that $H_1(E, \mathbb{Z})$ is cyclic. (2) and (3) and the remaining parts of (1) are just direct calculations. Finally to establish (4) we observe the following. Since the trefoil knot is 2-bridge E cannot contain a closed embedded essential surface by [HT]. If M contained an embedded incompressible surface $\neq T$, it would follow from the remark above and the gluing homeomorphism that E would have a boundary slope $1/0$. However [Theorem 2.0.3, CGLS] then implies the existence of a closed embedded essential surface in E . \square

To show that M is minimal, we assume not and suppose that there is a degree one map $f : M \rightarrow N$, where N is irreducible, $N \neq M$, and $N \neq S^3$. First, since M is a graph manifold, its Gromov norm is zero, so N cannot be a hyperbolic 3-manifold by [T, Chapter 6]. Moreover it is well-known that N must be an integer homology sphere ([Lemma 3.1 RW]. The proof of Theorem 4.1 will be finished by Lemmas 4.3, 4.4 and 4.5 below.

Lemma 4.3. *N is non-Haken.*

Proof. Suppose N is Haken, and let $F \subset N$ be an embedded incompressible surface. We may deform f so that $f^{-1}(F)$ is an incompressible surface in M . By (4) of Lemma 4.2 $f^{-1}(F)$ must consist of parallel copies of T . By standard 3-manifold topology, we can further deform f so that $f^{-1}(F) = T$. It follows that F is a 2-sphere or torus. Since N is irreducible, F must be a torus separating N into two parts N_1 and N_2 . Furthermore, the map f can be decomposed into two proper degree one map $f| : E_i \rightarrow N_i$. However E_i is a minimal 3-manifold among knot complements in 3-manifolds via proper degree one maps [BW]. Thus, each $f|$ is a homeomorphism, and it follows that f itself is homotopic to a homeomorphism. \square

Lemma 4.4. *N is not a Seifert manifold with finite fundamental group (other than possibly S^3).*

Proof. By (3) of Lemma 4.2 if N is a Seifert fibered manifold of finite fundamental group and $N \neq S^3$, it must be the Poincaré Homology 3-sphere P . Note $\pi_1(P)$ surjects onto A_5 , the alternating group on 5 letters. In particular, (as is well-known) A_5 is a subgroup of $PSL(2, \mathbb{C})$ —since $SO(3)$ can be identified with $PSU(2)$, and the latter is a

subgroup of $PSL(2, \mathbb{C})$. To prove the lemma, it suffices to prove that the image group of any representation of $\phi : \pi_1(M) \mapsto PSL(2, \mathbb{C})$ cannot be A_5 .

Case (1) If $\phi(t_1) \neq 1$ and $\phi(t_2) \neq 1$, by (1) of Lemma 4.2, the whole image $\phi(\pi_1(M))$ must be a cyclic group (actually trivial).

Case (2) Without loss of generality, we may assume that $\phi(t_1) = 1$ and $\phi(t_2) \neq 1$. By (1) and (2) of Lemma 4.2, $\phi : \pi_1(M) \rightarrow PSL(2, \mathbb{C})$ factors as $\nu : \pi_1(M) \rightarrow G$ and $\mu : G \rightarrow PSL(2, \mathbb{C})$ where G is generated by two groups described in (a) and (b) below:

(a) $\nu(\pi_1(E_1)) = \langle a_1, b_1, c_1 | a_1^2, b_1^3, a_1 b_1 c_1 \rangle$, (b) A cyclic group $\langle \lambda_2 \rangle$ such that $\nu(c_2) = \lambda_2^5$, $\nu(t_2) = \lambda_2^6$.

Since $h(t_1 c_1^{-1}) = t_2^5 c_2^{-6}$, we have $\nu(h(c_1^{-1})) = \nu(h(t_1 c_1^{-1})) = \nu(t_2^5 c_2^{-6}) = 1$. It follows that

$$G = \langle a_1, b_1, c_1 | a_1^2, b_1^3, a_1 b_1 c_1 \rangle = \langle a_1, b_1 | a_1^2, b_1^3, a_1 b_1 \rangle,$$

which is the trivial group.

Case (3) $\phi(t_1) = 1$ and $\phi(t_2) = 1$. In this case $\phi : \pi_1(M) \rightarrow PSL(2, \mathbb{C})$ factors through a group G via a map $\nu : \pi_1(M) \rightarrow G$, with $\nu(\pi_1(E_i))$ is the quotient of $G_i = \langle a_i, b_i, c_i | a_i^2, b_i^3, a_i b_i c_i \rangle$, $i = 1, 2$. Moreover by (2) of Lemma 4.2 we have that in the quotient $c_1 = c_2^6$ and $c_2 = c_1^{-6}$. Immediately we have that $c_1^{37} = 1$ and $c_2^{37} = 1$ and finally

$$G = \langle a_i, b_i, c_i, i = 1, 2 | a_i^2, b_i^3, a_i b_i c_i, c_i^{37}, c_1 = c_2^6, i = 1, 2 \rangle$$

Suppose there is a homomorphism $\mu : G \rightarrow A_5$. Since the order of c_i is 37, and A_5 has order 60, under the homomorphism μ the images of c_1 and c_2 must be trivial. It follows that $\mu : G \rightarrow A_5$ can factor through the group G' ,

$$G' = \langle a_1, b_1, | a_1^2, b_1^3, a_1 b_1 \rangle * \langle a_2, b_2, | a_2^2, b_2^3, a_2 b_2 \rangle,$$

but as above this is trivial. \square

Lemma 4.5. *N is not a Seifert manifold with infinite π_1 .*

The proof of Lemma 4.5 requires a sequence of additional lemmas. Suppose below N is a Seifert manifold of infinite π_1 . By Lemma 4.3, we may assume that N is non-Haken. Then N must be a Seifert manifold with three singular fibers over S^2 .

We begin by establishing:

Lemma 4.5.1. (1) *Suppose $\Delta \subset Iso_+ H^2$ is a triangle group and $\phi : \pi_1(2, 3, l) \rightarrow \Delta$ is of finite index. Then the image of ϕ is a hyperbolic triangle group isomorphic to $\pi_1(2, 3, k)$, where $k|l$.*

(2) *Suppose a Seifert manifold N is an integer homology sphere with infinite π_1 and orbifold $O = (a_1, a_2, a_3)$. Then $\gcd(a_i, a_j) = 1$ for $i, j = 1, 2, 3$, and O is a hyperbolic orbifold.*

Proof. (1) Let x', y' be the order 2 and order 3 elements which generates $\pi_1(2, 3, l)$ such that $x' y'$ is of order l . Use x and y to denote their images in $Iso_+ H^2$, then x and y generate the image of ϕ . Since the image of ϕ is of finite index in Δ , it must be co-compact and of rank 2. By well-know fact then the image is a triangle group with $x^2 = y^3 = (xy)^k = 1$, where $k|l$.

(2) follows from [p. 680 (d) LWZ]. \square

Lemma 4.5.2. *There is a simple closed curve in the kernel of $f| : T \rightarrow N$.*

Proof. Since $\pi_1(N)$ is torsion free and T is a torus, to prove the lemma, we need only that the kernel of $f| : T \rightarrow N$.

Suppose $f(t_1) \neq 1$, otherwise the claim is proved. Note that all elements in $f(\pi_1(E_1))$ commute with $f(t_1)$. If $f(t_1)$ is not the fiber t of N , then either

$$f(\pi_1(E_1)) = f(t_1) \text{ or } f(\pi_1(E_1)) = \langle f(t_1), f(c_1) \rangle = \mathbb{Z} \oplus \mathbb{Z}.$$

The second case is not possible since $H_1(E_1; \mathbb{Z}) = \mathbb{Z}$. In the first case we deduce that $\ker(f|_T)_*$ is nontrivial. Similarly if $f(t_2)$ is not the fiber t of N , then $\ker(f|_T)_*$ is non-trivial. If $f(t_1) = t = f(t_2)$. Since t_1 and t_2 do not coincide up to isotopy, still we have $\ker(f|_T)_*$ is non-trivial. \square

Let C be the simple closed curve provided by Lemma 4.5.2. Suppose $C = pm_1 + ql_1$ on ∂E_1 , then $C = -qm_2 + pl_2$. By (1) of Lemma 4.2 we have $pm + gl = (p - 5q)t + (-p + 6q)c$ and $-qm + pl = (-q - 5p)t + (q + 6p)c$. So the degree 1 map f factors through $f : M \rightarrow N_1 \cup_{S^1} N_2 \rightarrow N$ where N_1 and E'_2 are Seifert manifolds whose normal forms are given by $(2, 1; 3, 1; -p + 6q, p - 5q)$ and $(2, 1; 3, 1; q + 6p, -5p - q)$ respectively, and the two cores of the surgery solid tori are identified. If $f|_*(\pi_1(N_1)) \neq \pi_1(N)$ and $f|_*(\pi_1(N_2)) \neq \pi_1(N)$ then $\pi_1(N)$ can be presented as a non-trivial free product with amalgamation by the classical result (see [CGLS] for example). It follows that N will be Haken contrary to Lemma 4.3. Thus without loss, we assume that $f|_*(\pi_1(N_1)) = \pi_1(N)$.

Lemma 4.5.3. *$f|_{N_2}$ is of degree non-zero.*

Proof. Let \tilde{E} be the covering of N corresponding to $f|_*(\pi_1(N_2))$. Then $f : N_2 \rightarrow N$ lifts to $\tilde{f} : N_2 \rightarrow \tilde{E}$, which is π_1 -surjective. If $f|_*(\pi_1(N_2)) \subset \pi_1(N)$ is of finite index, then \tilde{E} is a closed Seifert manifold. Since both $\pi_1(N_1)$ and $\pi_1(N)$ are rank 2, $\pi_1(\tilde{E})$ must be also rank 2. Then \tilde{f} is of non-zero degree by Theorem 2.1. Hence $f|_{N_2}$ is non-zero degree.

Below we show $f|_*(\pi_1(N_2)) \subset \pi_1(N)$ must be of finite index. Otherwise \tilde{E} is a non-compact, aspherical Seifert manifold, which is known that either the rank of $H_1(\tilde{E})$ is positive or $\pi_1(\tilde{E})$ is trivial. Since $f|_*(\pi_1(N_2))$ is not trivial and N_2 is a rational homology sphere, all of the above cases are ruled out. So $f|_*(\pi_1(N_2))$ must be of finite index in $\pi_1(N)$. \square

Since N_1 and N_2 are in symmetry position, we have both $f|_{N_1}$ and $f|_{N_2}$ are of non-zero degree.

By Lemma 4.5.3, we may assume that $f|_{N_i}$ is fiber preserving. Then $f|_{N_i}$ induces an homomorphism $\phi_i : \pi_1(O_i) \rightarrow \pi_1(O)$, in particular ϕ_1 is surjective and ϕ_2 is finite index, where $O_1 = (2, 3, 6q - p)$, $O_2 = (2, 3, 6q + p)$ and $O = (a_1, a_2, a_3)$ are orbifolds of N_1 , N_2 and N respectively. Since N is an integer homology sphere of infinite π_1 , it follows that $\pi_1(O)$ is isomorphic to a hyperbolic triangle group. Since $\phi_1 : G_1 \rightarrow G$ is surjective, it follows that $O = (2, 3, k)$, where $k|6q - p$ by Lemma 4.5.1 (1). Since ϕ_2 is of finite index, the image of ϕ_2 is a hyperbolic triangle groups $\pi_1(2, 3, k')$ with $k'|6q + p$ by Lemma 4.5.1 (1), moreover $k'|k$. It is easy to see that k' is a divisor of both $12q$ and

2p. Since p and q are coprime, the great common divisor of $12q$ and $2p$ is 12. So k' is either 2, or 3, or 4, or 6, or 12. Then N can not be an integer homology sphere by 4.5.1 (2). \square

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